① (a)

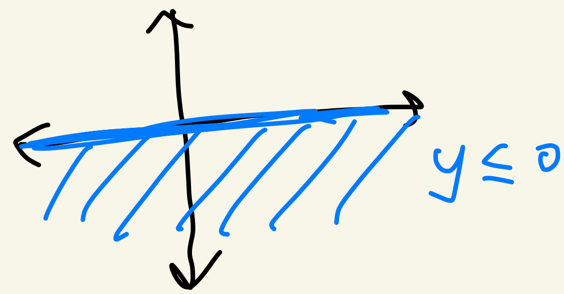
$$\frac{dy}{dx} = y^{2/3}, \quad y(0) = 2$$

$(x_0, y_0) = (0, 2)$

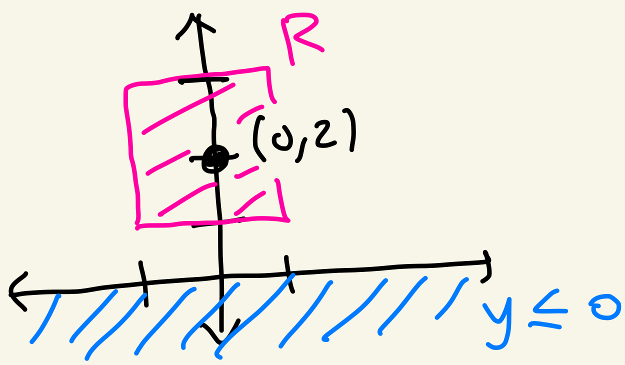
Here $f(x,y) = y^{2/3}$

$$\frac{\partial f}{\partial y} = \frac{2}{3} y^{-1/3} = \frac{2}{3} \cdot \frac{1}{y^{1/3}}$$

Note that f is continuous everywhere, but $\frac{\partial f}{\partial y}$ is not continuous when $y \le 0$



Let R be the rectangle defined by $-1 \le x \le 1$ and $1 \le y \le 3$.



Then, f and $\frac{\partial f}{\partial y}$ are continuous in R.

So, Picards theorem says that there exists a unique solution to the initial-value problem on some interval I around $x_0 = 0$.

$\boxed{(1)(b)}$

$$x\frac{dy}{dx} = y \,,\quad y(2) = 0$$

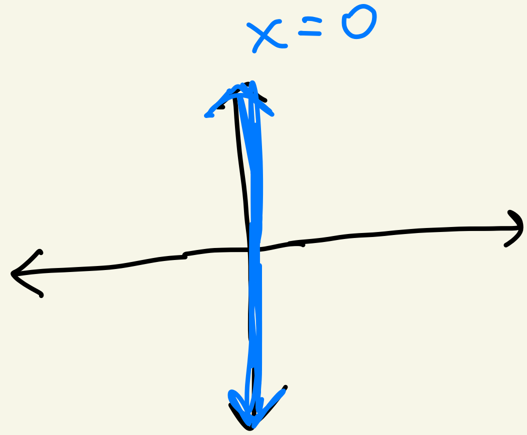
$\boxed{(x_0, y_0) = (2, 0)}$

This can be re-written as $\dfrac{dy}{dx} = \dfrac{y}{x}$

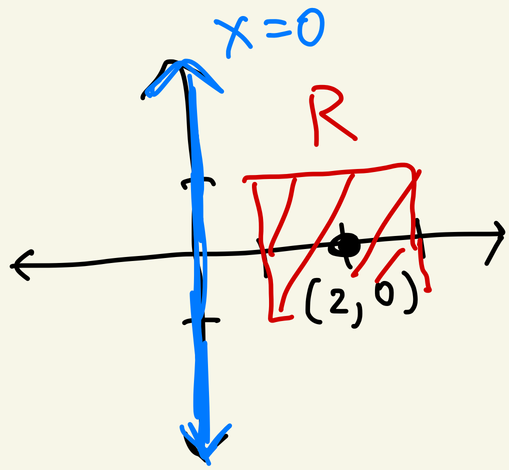
Let $f(x,y) = \dfrac{y}{x}$.

Then, $\dfrac{\partial f}{\partial y} = \dfrac{1}{x}$.

Note that f and $\dfrac{\partial f}{\partial y}$ are continuous everwhere except when $x = 0$.



Let R be the rectangle given by $1 \leq x \leq 3, -1 \leq y \leq 1$.



Then, f and $\dfrac{\partial f}{\partial y}$ are continuous in R.

Then, by Picard's theorem there exists a unique solution to the initial value problem on some interval containing $x_0 = 2$.

$\boxed{1(c)}$

$$y' - y = x, \quad y(1) = 2$$

We have

$$y' = x + y, \quad y(1) = 2$$

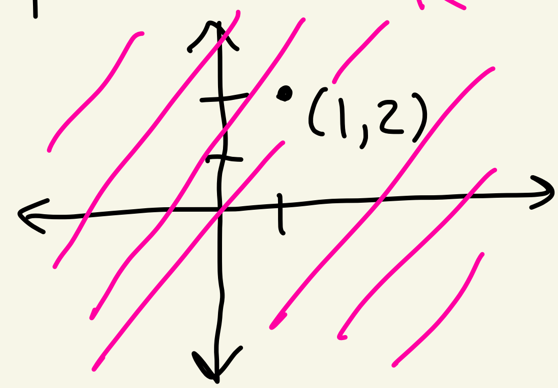
$\boxed{(x_0, y_0) = (1, 2)}$

Let $f(x,y) = x + y$.

Then, $\frac{\partial f}{\partial y} = 1$

f and $\frac{\partial f}{\partial y}$ are continuous everywhere.

Let R be the entire xy-plane.



Then, f and $\frac{\partial f}{\partial y}$ are continuous in R.

Then, by Picards theorem there exists a unique solution to the initial value problem on some interval I containing $x_0 = 1$.

1(d)

$$(4-y^2)\,y' = x^2,\quad y(0)=0$$

This can be re-written

$$y' = \frac{x^2}{4-y^2},\quad y(0)=0 \qquad \boxed{(x_0,y_0)=(0,0)}$$

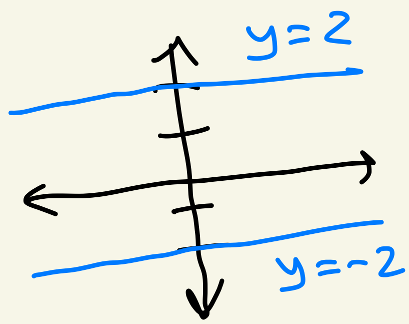
Then,

$$f(x,y) = \frac{x^2}{4-y^2} = x^2(4-y^2)^{-1}$$

and

$$\frac{\partial f}{\partial y} = -x^2(4-y^2)^{2}(-2y)$$

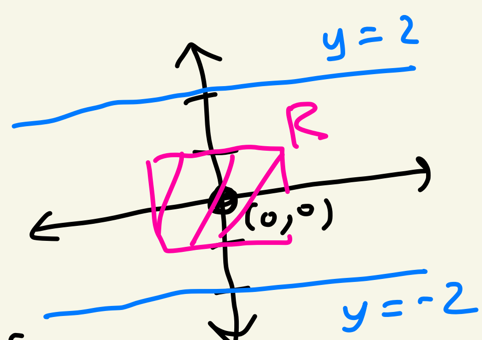
$$= \frac{2yx^2}{(4-y^2)^2}$$



f and $\frac{\partial f}{\partial y}$ are continuous everywhere except when $y=\pm 2$.

Let R be defined by

$-1 \le x \le 1,\ -1 \le y \le 1$



Then, f and $\frac{\partial f}{\partial y}$ are continuous on R. So, by Picards theorem there exists a unique solution to the initial value problem on some interval containing $x_0=0$.

$\boxed{②(a)}$

Let $y = cx$.

Then, $y' = c$

Thus, $xy' = xc = y$.

So, $y = cx$ solves $xy' = y$.

$\boxed{②(b)}$

Let $f_1(x) = x$ ← ($c=1$ from above)

Let $f_2(x) = 2x$ ← ($c=2$ from above)

Then from part (a) we know that f_1 and f_2 both solve $xy' = y$.

Further, $f_1(0) = 0$ and $f_2(0) = 2 \cdot 0 = 0$.

Thus, f_1 and f_2 both solve the initial value problem

$$xy' = y, \quad y(0) = 0.$$

Thus, this initial value problem does not have a unique solution.

3(a)

Let $y = cx^2$.

Then, $\frac{dy}{dx} = 2cx$

Thus, $x\frac{dy}{dx} = x(2cx) = 2cx^2 = 2y$

So, $y = cx^2$ solves $x\frac{dy}{dx} = 2y$

3(b)

Let $f_1(x) = 3x^2$ ← ($c = 3$ from above)

Let $f_2(x) = -x^2$ ← ($c = -1$ from above)

Then from part (a) we know that f_1 and f_2 both solve $x\frac{dy}{dx} = 2y$.

Further, $f_1(0) = 3(0)^2 = 0$ and $f_2(0) = -0^2 = 0$

Thus, f_1 and f_2 both solve the initial value problem

$$x\frac{dy}{dx} = 2y, \quad y(0) = 0.$$

Thus, this initial value problem does not have a unique solution.